ON HAMILTON'S PRINCIPLE FOR NONHOLONOMIC SYSTEMS

(O PRINTSIPE GAMIL'TONA DLIA NEGOLONOMNYKH SISTEM)

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There are two treatments of Hamilton's principle: the classical one, founded on the calculus of variations (Hertz [1], Kerner [2]) and the formal one, given by Holder [3]. Mechanical interpretations are known only for the first one of these, and one such interpretation will concern us here.

Hertz gave an example of a nonholonomic system (a sphere which rolls without slipping) to which Hamilton's principle is not applicable. In 1931, Kerner showed that, if one regards Hamilton's principle as a variational principle with side conditions when considering systems with differential constraints, then a necessary and sufficient condition that the Euler variational equations and the corresponding equations of motion coincide is that the system be holonomic. We shall consider below the possibility of deriving the equations of motion of systems with differential constraints from the variational principle with side conditions

$$\delta \int_{t_1}^{t_2} F dt = 0$$

where F is a certain function of time, the coordinates and the velocities of the particles of the system.

1. Consider a mechanical system. Let $g_1, \ldots, q_n$ be the Lagrangian coordinates, which are constrained only by the differential relations

$$\omega_\beta = q_\beta' + \sum_{\tau=1}^{n-m} a_{\beta, m+\tau}\, q_{m+\tau}' = 0 \qquad (\beta = 1, \ldots, m) \tag{1.1}$$

with coefficients differentiable continuously in a certain region A.

For simplicity it will be supposed that the constraints are scleronomic. Sometimes, for the sake of convenience, the constraints (1.1) will be written in the following form:

$$\omega_\beta = \sum_{s=1}^{n} a_{\beta s} q_s' = 0, \qquad a_{\beta s} = \begin{cases} 1 & \text{for } s = \beta \\ 0 & \text{for } m > s \neq \beta \\ a_{\beta,\, m+\tau} & \text{for } s > m \end{cases} \tag{1.2}$$

A motion

$$q_s = \varphi_s(t) \qquad (s = 1, \ldots, n) \tag{1.3}$$

of a material system will be said to be kinematically admissible provided that the functions $\phi_s(t)$ satisfy the constraints identically.

Generalizing Hamilton's principle, let us replace the Lagrangian function L in the integrand by an arbitrary differentiable function F depending upon time, the coordinates and the velocities of the particles of the system. Thus we are led to the variational principle with side conditions

$$\delta \int_{t_1}^{t_2} F dt = 0 \tag{1.4}$$

where the variation is to be taken over the kinematically admissible motions; here, as usual

$$\delta q_s(t_1) = \delta q_s(t_2) = 0 \qquad (s = 1, \ldots. n)$$

Euler's equations for the variational principle (1.4), with constraints (1.2), using Lagrange multipliers λ_β, are

$$\frac{d}{dt}\frac{\partial F}{\partial q_s'} - \frac{\partial F}{\partial q_s} + \sum_{\beta=1}^{m} \lambda_\beta' a_{\beta s} + \sum_{\beta=1}^{m} \lambda_\beta \left(a_{\beta s}' - \frac{\partial \omega_\beta}{\partial q_s}\right) = 0 \tag{1.5}$$

or, carrying out the differentiations, to put the acceleration terms in evidence

$$\sum_{r=1}^{n} \frac{\partial^2 F}{\partial q_s' \partial q_r'} q_r'' + \sum_{\beta=1}^{m} \lambda_\beta' a_{\beta s} + \sum_{\beta=1}^{m} \lambda_\beta \left(a_{\beta s}' - \frac{\partial \omega_\beta}{\partial q_s}\right) + \ldots = 0 \tag{1.6}$$

The constraint equations (1.1) yield

$$q_\beta'' + \sum_{\tau=1}^{n-m} a_{\beta,\, m+\tau}\, q_{m+\tau}'' + \ldots = 0 \tag{1.7}$$

In Equations (1.6) and (1.7) the dots denote terms which do not contain q'', λ' and λ . Let us suppose that we are in the "normal" case, i.e. when the determinant Δ

$$
\begin{vmatrix}
\dfrac{\partial^2 F}{\partial q_1'^2} & \cdots & \dfrac{\partial^2 F}{\partial q_1' \partial q_m'} & \dfrac{\partial^2 F}{\partial q_1' \partial q_{m+1}'} & \cdots & \dfrac{\partial^2 F}{\partial q_1' \partial q_n'} & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\dfrac{\partial^2 F}{\partial q_m' \partial q_1'} & \cdots & \cdots & \cdots & \cdots & \dfrac{\partial^2 F}{\partial q_m' \partial q_n'} & 0 & \cdots & 1 \\
\dfrac{\partial^2 F}{\partial q_{m+1}' \partial q_1'} & \cdots & \cdots & \cdots & \cdots & \dfrac{\partial^2 F}{\partial q_{m+1}' \partial q_n'} & a_{1,\, m+1} & \cdots & a_{m,\, m+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\dfrac{\partial^2 F}{\partial q_n' \partial q_1'} & \cdots & \dfrac{\partial^2 F}{\partial q_n' \partial q_m'} & \dfrac{\partial^2 F}{\partial q_n' \partial q_{m+1}'} & \cdots & \dfrac{\partial^2 F}{\partial q_n'^2} & a_{1n} & \cdots & a_{mn} \\
1 & \cdots & 0 & a_{1,\, m+1} & \cdots & a_{1n} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 1 & a_{m,\, m+1} & \cdots & a_{mn} & 0 & \cdots & 0
\end{vmatrix} \neq 0 \qquad (1.8)
$$

Under these assumptions Equations (1.6) and (1.7) may be solved for the q_r'' and λ_β', and thus

$$q_r'' = Q_r(t, q, q', \lambda), \qquad \lambda_\beta' = \Lambda_\beta(t, q, q', \lambda) \qquad (1.9)$$

2. In classical mechanics the following principle 1 always holds: the acceleration of a particle of a system may be uniquely determined at a given moment provided that at this same moment one knows the external forces, the constraints and all the coordinates and velocities of the particle in question.

V.I. Kirgetov suggested the use of this principle as a criterion for the applicability of the variational principle (1.4).

Consider Equations (1.9). Let us assume that in the kinematically admissible motion the functions Q depend upon the parameters λ_β for whose determination only differential equations are available. The parameters λ_β may only be determined by means of the integration of the system of equations (1.9). In other words, in order to know the dependence of the accelerations on the instantaneous position coordinates and velocities it is necessary to know at all times during the motion the dependence of the accelerations on the coordinates, on the velocities and on the parameters λ_β. This contradicts the mechanical principle 1, although it does not interfere with the solution of the mathematical problem.

In order that principle 1 hold it is necessary (and clearly also sufficient) that for kinematically admissible motions the functions Q_r be independent of the parameters λ_β, i.e. that all the partial derivatives

$\partial Q_r / \partial \lambda_\beta$ vanish identically in t, q, q' as a consequence of the constraints

$$\partial Q_r / \partial \lambda_\beta = 0 \qquad (r = 1, \ldots, n;\ \beta = 1, \ldots, m) \tag{2.1}$$

Theorem. For nonholonomic systems the Euler equations of the variational principle (1.4), with fixed end-points and with the side conditions expressed by the constraints (1.1)

$$\omega_\beta = 0$$

are never compatible, generally speaking, with principle 1, for any function F.

The phrase "generally speaking" refers to the particular case $\Delta = 0$ in the variational problem.

Proof. Let us make evident the dependence of Q_r on λ_β. It is obvious that

$$Q_r = \frac{\Delta_r}{\Delta}$$

where Δ_r is obtained from the determinant Δ by replacing the rth column by the free terms of Equations (1.6) and (1.7). The determinant Δ does not depend upon λ_β, therefore (2.1) holds identically in t, q, q' in view of the constraint equations. Differentiating, we find that the determinant $\partial \Delta_r / \partial \Delta_\beta = \Delta_{r\beta}$ equals zero, a determinant which is obtained from Δ by replacing the rth column of Δ by the following elements, written in succession from top to bottom:

$$a_{\beta 1}' - \frac{\partial \omega}{\partial q_1}, \ldots, \ a_{\beta n}' - \frac{\partial \omega_\beta}{\partial q_n}, \qquad 0, \ldots, 0$$

Lemma. Suppose that the determinant of order $n + m$

$$D = \begin{vmatrix} f_{11} & \ldots f_{1n} & f_{1,\,n+1} & \ldots f_{1,\,n+} \\ \cdot & \cdot & \cdot & \cdot \\ f_{n+m,\,1} & \ldots f_{n+m,\,n} & f_{n+m,\,n+1} & \ldots f_{n+m,\,n+m} \end{vmatrix}$$

is not zero, and that when the first n columns are replaced, in succession, by the column

$$d = \begin{pmatrix} d_1 \\ \cdots \\ d_{n+m} \end{pmatrix}$$

while leaving the other columns of D unaltered, one obtains n determinants $D_r (r = 1, \ldots, n)$ which are all zero. Then the column d must be a linear combination of the last m columns of the determinant D, that is

$$d_j = \sum_{\rho=1}^{m} c_{n+\rho} f_{j,\,n+\rho} \qquad (j = 1, \ldots, n+m)$$

Indeed, since $D \neq 0$ and $D_r = 0$, the column d, which is the rth column of the determinant D_r, must be a linear combination of the remaining columns of the determinant D_r

$$d_j = \sum_{\substack{s=1 \\ s \neq r}}^{n} b_s^{(r)} f_{js} + \sum_{\rho=1}^{m} c_{n+\rho}^{(r)} f_{j,\,n+\rho} \qquad (j = 1, \ldots, n+m)$$

Analogously, for $l < n$

$$d_j = \sum_{\substack{t=1 \\ t \neq l}}^{n} b_t^{(l)} f_{jt} + \sum_{\rho=1}^{m} c_{n+\rho}^{(l)} f_{j,\,n+\rho}$$

from which it follows that

$$\sum_{\substack{k=1 \\ k\neq r,\, k\neq l}}^{n} f_{jk} \left(b_k^{(r)} - b_k^{(l)}\right) + \sum_{\rho=1}^{m} f_{j,\,n+\rho} \left(c_{n+\rho}^{(r)} - c_{n+\rho}^{(l)}\right) + b_l^{(r)} f_{jl} - b_r^{(l)} f_{jr} = 0$$

and since $D \neq 0$, these equations give

$$b_k^{(r)} = b_k^{(l)}, \qquad b_l^{(r)} = b_r^{(l)} = 0, \qquad c_{n+\rho}^{(r)} = c_{n+\rho}^{(l)} = c_{n+\rho}$$

for arbitrary l, $r = 1, \ldots, n$, and the lemma is proved.

In view of the lemma, applied to the determinant Δ, we have

$$a_{\beta s}' - \frac{\partial \omega_\beta}{\partial q_s} = \sum_{\rho=1}^{m} c_{n+\rho}\, a_{\rho s} \qquad (s = 1, \ldots, n) \tag{2.2}$$

which, together with (1.2), yields

$$a_{\beta,\,m+\sigma}' - \frac{\partial \omega_\beta}{\partial q_{m+\sigma}} = \sum_{\rho=1}^{m} a_{\rho,\,m+\sigma} \left(a_{\beta\rho}' - \frac{\partial \omega_\beta}{\partial q_\rho}\right) \qquad \begin{pmatrix} \beta = 1, \ldots, m \\ \sigma = 1, \ldots, n-m \end{pmatrix}$$

As a consequence of these identities, and of (1.1) and (1.2), we obtain

$$\sum_{\sigma=1}^{n-m} \left[\frac{\partial a_{\beta,\,m+\sigma}}{\partial q_{m+\tau}} - \frac{\partial a_{\beta,\,m+\tau}}{\partial q_{m+\sigma}} + \sum_{\rho=1}^{m} \left(a_{\rho,\,m+\sigma} \frac{\partial a_{\beta,\,m+\tau}}{\partial q_\rho} - a_{\rho,\,m+\tau} \frac{\partial a_{\beta,\,m+\sigma}}{\partial q_\rho}\right)\right] q_{m+\sigma}' = 0$$

The last equation must hold for arbitrary $q_{m+\sigma}'$ and an arbitrary point of the domain A. Hence

$$\frac{\partial a_{\beta, m+\sigma}}{\partial q_{m+\tau}} - \frac{\partial a_{\beta, m+\tau}}{\partial q_{m+\sigma}} + \sum_{\rho=1}^{m} \left(a_{\rho, m+\sigma} \frac{\partial a_{\beta, m+\tau}}{\partial q_\rho} - a_{\rho, m+\tau} \frac{\partial a_{\beta, m+\sigma}}{\partial q_\rho} \right) = 0 \qquad (2.3)$$

$$(\beta = 1, \ldots, m; \ \sigma, \tau = 1, \ldots, n-m)$$

must hold throughout the domain A.

These equations are the necessary and sufficient conditions for the integrability of the constraint relations (1.1), and the theorem is proved.

3. The fundamental equations (2.1) may be proved in another manner. Instead of principle 1, let us employ the following principle 2 which is equivalent to it: the motion of the particles of the system under given forces is uniquely determined by the initial values of time, all of its coordinates and all of its velocities. According to this principle Equations (1.9) define a family of motions which depend on $2n + m + 1$ parameters t_0, q_{s0}, q_{s0}', $\lambda_{\beta 0}$:

$$q_r = \varphi_r(t;\ t_0, q_{s0}, q_{s0}, \lambda_{\beta 0})$$

which obey Equations (1.1)

$$_\beta = q_{\beta 0}' + \sum_{\tau=1}^{n-m} a_{\beta, m+\tau}(q_{s0})\, q_{m+\tau, 0}' = 0$$

For fixed t_0, q_{s0}, q_{s0}' one obtains an m-parameter family of motions depending on the parameters $\lambda_{\beta 0}$. Hence it is obvious that the necessary and sufficient condition for the validity of principle 2 is that the functions ϕ_r be entirely independent of the parameters $\lambda_{\beta 0}$, which means that (2.1) holds.

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